Valuation of Interest-Bearing and Non-Interest-Bearing Liabilities
Part II - Company Revenue Growth Rate is Time-Dependent

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In Part I we built a model to value interest-bearing (IBL) and non-interest-bearing liabilities (NIBL) where the revenue growth rate was constant. In Part II we will change the assumption of a constant revenue growth rate to a revenue growth rate that is mean-reverting and time-dependent. It should be noted that when modeling mean reversion we use the incomplete gamma function, which is applicable only to cases where the revenue growth rate is decreasing over time. Note that a decreasing revenue growth rate should be far more common than an increasing rate for established companies that are not start-ups.

We will work through the following hypothetical problem from Part I but will assume a revenue growth rate that is mean-reverting.

Our Hypothetical Problem

We are currently standing at time zero and are tasked with determining the value of a company's liabilities. These liabilities can be either interest-bearing or non-interest-bearing. Our go-forward model assumptions are...

Description	Value
Annualized revenue (in dollars)	1,000,000
Annualized short-term revenue growth rate (percent)	12.00
Annualized long-term revenue growth rate (percent)	3.00
Excess revenue growth rate half-life (years)	5.00
Ratio of assets to annualized revenue	1.20
Ratio of total liabilities to total assets	0.45
Annualized debt interest accrual rate (percent)	4.00
Annualized debt coupon payment rate (percent)	2.00
Annualized debt risk-adjusted discount rate (percent)	6.00

Question 1: What is the balance of total liabilities at time zero?

Question 2: What is the market value of IBL at time zero given that term is 15 years?

Question 3: What is the market value of IBL given that term is perpetual?

Question 4: What is the market value of NIBL given that term is perpetual?

IBL Equations

In Part I we defined the variable μ to be the continuous-time revenue growth rate that was constant over time. In Part II of the series we will add a time subscript to that variable that will allow the growth rate to change over time. We will define the variable λ to be the rate of mean reversion and the variable W_t to be the value of the driving Brownian motion at time t. The stochastic differential equation that defines the evolution of the revenue growth rate over time is... [1]

$$\delta \mu_t = \lambda \bigg(\mu_\infty - \mu_t \bigg) \delta t + \sigma \, \delta W_t \; ...\text{where...} \; 0 < \lambda < 1 \tag{1}$$

The revenue growth rate rate transitions from the unsustainable rate (μ_0) to the sustainable rate (μ_∞) over the time interval $[0, \infty]$. To calibrate the rate of mean reversion we will choose some future point in time ($t = h$) where the expected short rate at time h is exactly halfway between the short-term rate and the long-term rate (i.e. h is the half life). The equation to calibrate the rate of mean reversion is... [1]

$$\text{Exp}\left\{-\lambda \times h\right\} = 0.50 \text{ ...such that... } \lambda = -\frac{\ln(0.50)}{h} \tag{2}$$

The revenue growth rate at time t is equal to the revenue growth rate at time zero plus the cumulative changes in that rate over the time interval $[0, t]$. Using Equation (1) above the equation for the random revenue growth rate at time t is... [1]

$$\mu_t = \mu_0 + \int_0^t \delta\mu_s \tag{3}$$

The equation for the mean of the revenue growth rate (RGR) at time t is... [1]

$$\mu_t = \mu_\infty + \Delta \text{Exp}\left\{-\lambda t\right\} \text{ ...where... } \Delta = \mu_0 - \mu_\infty \text{ ...and... } \mu_0 > \mu_\infty \tag{4}$$

The equations for the short-term unsustainable and long-term sustainable revenue growth rates are...

$$\mu_0 = \ln\left(1 + \text{annualized short-term RGR}\right) \bigg| \mu_\infty = \ln\left(1 + \text{annualized long-term RGR}\right) \tag{5}$$

We will define the variable Γ_t to be the cumulative revenue rate over the time interval $[0, t]$. Using Equation (4) above the equation for the cumulative revenue growth rate is... [1]

$$\Gamma_t = \int_0^t \mu_s \, \delta s = \mu_\infty t + \Delta\left(1 - \text{Exp}\left\{-\lambda t\right\}\right)\lambda^{-1} \tag{6}$$

In Part I we defined the variable R_t to be annualized revenue at time t. Using Equation (6) above the equation for annualized revenue at time t as a function of annualized revenue at time zero is...

$$R_t = R_0 \text{Exp}\left\{\Gamma_t\right\} = R_0 \text{Exp}\left\{\mu_\infty t + \Delta\left(1 - \text{Exp}\left\{-\lambda t\right\}\right)\lambda^{-1}\right\} \tag{7}$$

In Part I we defined the variable A_t to be total assets at time t and the variable θ to be the ratio of total assets to annualized revenue. Using Equation (7) above the equation for total assets at time t as a function of annualized revenue at time zero is...

$$A_t = \theta R_t = \theta R_0 \text{Exp}\left\{\mu_\infty t + \Delta\left(1 - \text{Exp}\left\{-\lambda t\right\}\right)\lambda^{-1}\right\} \tag{8}$$

In Part I we defined the variable P_t to be debt principal balance at time t and the variable ϕ to be the ratio of debt principal to total assets. Using Equation (8) above the equation for debt principal balance outstanding is...

$$P_t = \phi A_t = \phi \theta R_0 \text{Exp}\left\{\mu_\infty t + \Delta\left(1 - \text{Exp}\left\{-\lambda t\right\}\right)\lambda^{-1}\right\} \tag{9}$$

In Part I we defined the variable ω to be the continuous-time debt interest accrual rate and the variable α to be the continuous-time debt coupon payment (i.e. debt service) rate. The equations for these two variables from Part I are...

$$\omega = \ln\left(1 + \text{annualized debt interest accrual rate}\right) \bigg| \alpha = \ln\left(1 + \text{annualized debt coupon payment rate}\right) \tag{10}$$

In Part I we defined the variable I_t to be accrued interest payable at time t. Using Equations (7), (8) and (10) above the equation for accrued interest payable at time t is...

$$I_t = \int_0^t (\omega - \alpha)\,\phi\,\theta R_0 \text{Exp}\left\{\Gamma_s\right\} \text{Exp}\left\{\omega(t - s)\right\} \delta s \text{ ...when... } I_0 = 0 \tag{11}$$

Using Appendix Equation (40) below the solution to Equation (11) above is..

$$I_t = (\omega - \alpha)\,\phi\,\theta R_0 \text{Exp}\left\{d\right\} a^{\frac{c}{b}} b^{-1}\left[\Gamma\left(-\frac{c}{b}, a \text{Exp}\left\{-b t\right\}\right) - \Gamma\left(-\frac{c}{b}, a\right)\right] \text{ ...when... } I_0 = 0 \tag{12}$$

IBL Valuation Equations

In Part I we defined the variable T to be debt maturity time in years and the variable C_t to be expected cash flow to be received over the time interval $[t, t + \delta t]$ where $t < T$ (i.e. debt service). Using Equations (4) and (6) above the equation for debt service cash flows where the revenue growth rate is time dependent is...

$$C_t = (\alpha - \mu_t)\, \phi\, \theta R_0 \operatorname{Exp}\left\{\Gamma_t\right\} \delta t \tag{13}$$

Using Appendix Equation (41) below the solution to Equation (13) above is...

$$C_t = \phi\, \theta R_0 \left[(\alpha - \mu_\infty) \operatorname{Exp}\left\{\mu_\infty t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda}\operatorname{Exp}\left\{-\lambda t\right\}\right\} - \Delta \operatorname{Exp}\left\{(\mu_\infty - \lambda) t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda}\operatorname{Exp}\left\{-\lambda t\right\}\right\}\right] \delta t \tag{14}$$

In Part I we defined the variable κ to be the continuous-time risk-adjusted discount rate. The equation for the discount rate is...

$$\kappa = \ln\left(1 + \text{annualized risk-adjusted discount rate}\right) \tag{15}$$

In Part I we defined the variable $PVIBL(0, T)$ to be the present value at time zero of debt service payments over the time interval $[0, T]$. Using Equations (14) and (15) above the equation for the present value of expected debt service is...

$$PVIBL(0, T) = \int_0^T C_t \operatorname{Exp}\left\{-\kappa t\right\} = \int_0^T \phi\, \theta R_0 \left[(\alpha - \mu_\infty) \operatorname{Exp}\left\{(\mu_\infty - \kappa) t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda}\operatorname{Exp}\left\{-\lambda t\right\}\right\}\right.$$
$$\left. - \Delta \operatorname{Exp}\left\{(\mu_\infty - \kappa - \lambda) t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda}\operatorname{Exp}\left\{-\lambda t\right\}\right\}\right] \delta t \tag{16}$$

We will make the following definitions...

$$a = \frac{\Delta}{\lambda} \ \text{...and...} \ b = \lambda \ \text{...and...} \ c_1 = \mu_\infty - \kappa \ \text{...and...} \ c_2 = \mu_\infty - \kappa - \lambda \ \text{...and...} \ d = \frac{\Delta}{\lambda} \ \text{...and...} \ e = \alpha - \mu_\infty \tag{17}$$

Using the definitions in Equation (17) above we can rewrite Equation (16) above as...

$$PVIBL(0, T) = \phi\, \theta R_0 \left[e \int_0^T \operatorname{Exp}\left\{d + c_1 t + a \operatorname{Exp}\left\{-b t\right\}\right\} \delta t - \Delta \int_0^T \operatorname{Exp}\left\{d + c_2 t + a \operatorname{Exp}\left\{-b t\right\}\right\} \delta t\right] \tag{18}$$

Using Appendix Equation (39) below the solution to Equation (18) above is...

$$PVIBL(0, T) = \phi \theta R_0 \left[e \operatorname{Exp}\left\{d\right\} a^{\frac{c_1}{b}} b^{-1}\left[\Gamma\left(-\frac{c_1}{b}, a \operatorname{Exp}\left\{-b n\right\}\right) - \Gamma\left(-\frac{c_1}{b}, a \operatorname{Exp}\left\{-b m\right\}\right)\right]\right.$$
$$\left. - \Delta \operatorname{Exp}\left\{d\right\} a^{\frac{c_2}{b}} b^{-1}\left[\Gamma\left(-\frac{c_2}{b}, a \operatorname{Exp}\left\{-b n\right\}\right) - \Gamma\left(-\frac{c_2}{b}, a \operatorname{Exp}\left\{-b m\right\}\right)\right]\right] \tag{19}$$

In Part I we defined the variable $PVIBL(T)$ to be the present value of debt principal plus accrued interest at debt maturity. Using Equations (9) and (12) above the equation for the present value of expected cash flow received at debt maturity is...

$$PVIBL(T) = \left(P_T + I_T\right) \operatorname{Exp}\left\{-\kappa T\right\} \tag{20}$$

Using Equations (19) and (20) the equation for the value of interest-bearing liabilities where $T < \infty$ is...

$$PVIBL = PVIBL(0, T) + PVIBL(T) \ \text{...where...} \ T < \infty \tag{21}$$

Using Appendix Equation (40) below and Equation (18) above the value of debt service when debt is perpetual (i.e. term is infinity) is...

$$PVIBL(0, \infty) = \phi \theta R_0 \left[e \operatorname{Exp}\left\{d\right\} a^{\frac{c_1}{b}} b^{-1}\left[\Gamma\left(-\frac{c_1}{b}, 0\right)\right) - \Gamma\left(-\frac{c_1}{b}, a\right)\right]\right.$$
$$\left. - \Delta \operatorname{Exp}\left\{d\right\} a^{\frac{c_2}{b}} b^{-1}\left[\Gamma\left(-\frac{c_2}{b}, 0\right)\right) - \Gamma\left(-\frac{c_2}{b}, a\right)\right]\right] \tag{22}$$

Using Equation (20) above the equation for the present value of expected cash flow received at debt maturity when debt is perpetual (i.e. term is infinity) is...

$$PVIBL(\infty) = 0 \text{ ...because... } \lim_{T \to \infty} \text{Exp}\left\{ -\kappa T \right\} = 0 \tag{23}$$

Using Equations (22) and (23) above the equation for the value of interest-bearing liabilities where $T = \infty$ is...

$$PVIBL = PVIBL(0, \infty) \tag{24}$$

NIBL Valuation Equations

For non-interest-bearing liabilities (NIBL) the interest accrual rate (ω) is zero, the coupon rate (α) is zero, and the term is infinite. We will make the following definitions...

$$a = \frac{\Delta}{\lambda} \text{ ...and... } b = \lambda \text{ ...and... } c_1 = \mu_\infty - \kappa \text{ ...and... } c_2 = \mu_\infty - \kappa - \lambda \text{ ...and... } d = \frac{\Delta}{\lambda} \text{ ...and... } e = -\mu_\infty \tag{25}$$

Using Equation (22) and the definitions in Equatino (25) above the equation for the value of non-interest-bearing liabilities is...

$$PVNIBL = \phi\,\theta R_0 \left[e\,\text{Exp}\left\{ d \right\} a^{\frac{c_1}{b}} b^{-1} \left[\Gamma\left(-\frac{c_1}{b}, 0 \right\} \right) - \Gamma\left(-\frac{c_1}{b}, a \right) \right]$$
$$- \Delta\,\text{Exp}\left\{ d \right\} a^{\frac{c_2}{b}} b^{-1} \left[\Gamma\left(-\frac{c_2}{b}, 0 \right\} \right) - \Gamma\left(-\frac{c_2}{b}, a \right) \right] \tag{26}$$

The Answers To Our Hypothetical Problem

Using Equation (5) above the equations for the continuous-time short-term and long-term revenue growth rates are...

$$\mu_0 = \ln\left(1 + 0.12 \right) = 0.11333 \text{ ...and.... } \mu_\infty = \ln\left(1 + 0.03 \right) = 0.02956 \tag{27}$$

Using Equation (2) above the equation for the rate of mean reversion is...

$$\lambda = -\frac{\ln(0.50)}{5.00} = 0.13863 \tag{28}$$

Using Equation (10) above the equation for the continuous-time debt interest accrual rate is...

$$\omega = \ln\left(1 + 0.04 \right) = 0.03922 \tag{29}$$

Using Equation (10) above the equation for the continuous-time debt coupon payment rate is...

$$\alpha = \ln\left(1 + 0.02 \right) = 0.01980 \tag{30}$$

Using Equation (15) above the equation for the continuous-time debt risk-adjusted discount rate is...

$$\kappa = \ln\left(1 + 0.06 \right) = 0.05827 \tag{31}$$

Question 1:What is the balance of total liabilities at time zero?

Using Equation (9) above and the model assumptions above the answer to the question is...

$$\text{IBL/NIBL balance at time zero} = 0.45 \times 1.20 \times 1,000,000 = 540,000 \tag{32}$$

Question 2: What is the market value of IBL at time zero given that term is 15 years?

The answer to the question is...

Description	Value	Reference	Notes
PV of debt service	-412,456	Equation (19)	
PV of principal payment	595,651	Equations (9), (31)	$1,427,513 \times \text{Exp}\left[-0.05827 \times 15\right]$
PV of accrued interest	156,880	Equations (12), (31)	$375,972 \times \text{Exp}\left[-0.05827 \times 15\right]$
Total	340,075		

Question 3: What is the market value of IBL given that term is perpetual?

The answer to the question is...

Description	Value	Reference
PV of debt service	-666,557	Equation (22)
PV of principal payment	0	
PV of accrued interest	0	
Total	-666,557	

Question 4: What is the market value of NIBL given that term is perpetual?

The answer to the question is...

Description	Value	Reference
PV of debt service	-1,287,698	Equation (26)
PV of principal payment	0	
PV of accrued interest	0	
Total	-1,287,698	

References

[1] Gary Schurman, *Mean Reversion Equations,* October, 2021.

[2] Gary Schurman, *The Incomplete Gamma Function - Part II,* December, 2017.

Appendix

A. The base integral equation for a mean-reverting process is... [2]

$$f(t) = \int_{m}^{n} \text{Exp}\left\{d + ct - a\,\text{Exp}\left\{-bt\right\}\right\}\delta t \text{ ...where... } a > 0 \;,\; b > 0 \;,\; c < 0 \;,\; n > m \geq 0 \tag{33}$$

The solution to Equation (33) above where $\Gamma(x,y)$ is the incomplete gamma function is... [2]

$$f(t) = \text{Exp}\left\{d\right\} a^{\frac{c}{b}} b^{-1}\left[\Gamma\left(-\frac{c}{b}, a\,\text{Exp}\left\{-bn\right\}\right) - \Gamma\left(-\frac{c}{b}, a\,\text{Exp}\left\{-bm\right\}\right)\right] \tag{34}$$

Note that when $m = 0$ and $n = \infty$ Equation (34) above becomes...

$$f(t) = \text{Exp}\left\{d\right\} a^{\frac{c}{b}} b^{-1}\left[\Gamma\left(-\frac{c}{b}, 0\right) - \Gamma\left(-\frac{c}{b}, a\right)\right] \tag{35}$$

The solution to the upper incomplete gamma function using standard Excel functions is... [2]

$$\Gamma(\alpha, x) = \text{EXP(GAMMALN(alpha))} \times (1 - \text{GAMMA.DIST(x,alpha,1,true)}) \tag{36}$$

B. The solution to Equation (11) above is...

$$I_t = \int_0^t (\omega - \alpha)\, \phi\, \theta R_0 \operatorname{Exp}\left\{\Gamma_s\right\} \operatorname{Exp}\left\{\omega\,(t-s)\right\} \delta s$$

$$= (\omega - \alpha)\, \phi\, \theta R_0 \operatorname{Exp}\left\{\omega\, t\right\} \int_0^t \operatorname{Exp}\left\{\Gamma_s - \omega\, s\right\} \delta s$$

$$= (\omega - \alpha)\, \phi\, \theta R_0 \operatorname{Exp}\left\{\omega\, t\right\} \int_0^t \operatorname{Exp}\left\{\left(\mu_\infty - \omega\right) s + \Delta\left(1 - \operatorname{Exp}\left\{-\lambda s\right\}\right)\lambda^{-1}\right\} \delta s \tag{37}$$

We will make the following parameter definitions...

$$a = \frac{\Delta}{\lambda} \ \ \text{...and...}\ \ b = \lambda \ \ \text{...and...}\ \ c = \mu_\infty - \omega \ \ \text{...and...}\ \ d = \frac{\Delta}{\lambda} \tag{38}$$

Using the parameter definitions in Equation (38) above we can rewrite Equation (37) above as...

$$I_t = (\omega - \alpha)\, \phi\, \theta R_0 \int_0^t \operatorname{Exp}\left\{d + c\, t - a \operatorname{Exp}\left\{-b\, t\right\}\right\} \delta t \ \ \text{...where...}\ \ a > 0 \ , \ b > 0 \ , \ c < 0 \tag{39}$$

Using Equation (34) above the solution to Equation (39) above is...

$$I_t = (\omega - \alpha)\, \phi\, \theta R_0 \operatorname{Exp}\left\{d\right\} a^{\frac{c}{b}} b^{-1}\left[\Gamma\left(-\frac{c}{b}, a \operatorname{Exp}\left\{-b\, t\right\}\right) - \Gamma\left(-\frac{c}{b}, a\right)\right] \tag{40}$$

C. Using Equations (4) and (6) above the solution to Equation (13) above is...

$$C_t = (\alpha - \mu_t)\, \phi\, \theta R_0 \operatorname{Exp}\left\{\Gamma_t\right\} \delta t$$

$$= \phi\, \theta R_0\left(\alpha - \mu_\infty - \Delta \operatorname{Exp}\left\{-\lambda t\right\}\right) \operatorname{Exp}\left\{\mu_\infty t + \Delta\left(1 - \operatorname{Exp}\left\{-\lambda t\right\}\right)\lambda^{-1}\right\} \delta t$$

$$= \phi\, \theta R_0\left(\alpha - \mu_\infty - \Delta \operatorname{Exp}\left\{-\lambda t\right\}\right) \operatorname{Exp}\left\{\mu_\infty t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda} \operatorname{Exp}\left\{-\lambda t\right\}\right\} \delta t$$

$$= (\alpha - \mu_\infty)\, \phi\, \theta R_0 \operatorname{Exp}\left\{\mu_\infty t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda} \operatorname{Exp}\left\{-\lambda t\right\}\right\} \delta t - \Delta\, \phi\, \theta R_0 \operatorname{Exp}\left\{(\mu_\infty - \lambda) t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda} \operatorname{Exp}\left\{-\lambda t\right\}\right\} \delta t$$

$$= \phi\, \theta R_0\left[(\alpha - \mu_\infty) \operatorname{Exp}\left\{\mu_\infty t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda} \operatorname{Exp}\left\{-\lambda t\right\}\right\} - \Delta \operatorname{Exp}\left\{(\mu_\infty - \lambda) t + \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda} \operatorname{Exp}\left\{-\lambda t\right\}\right\}\right] \delta t$$

$$\tag{41}$$